

Representability of Hom implies flatness

Nitin Nitsure

Abstract

Let X be a projective scheme over a noetherian base scheme S , and let \mathcal{F} be a coherent sheaf on X . For any coherent sheaf \mathcal{E} on X , consider the set-valued contravariant functor $\mathfrak{H}om_{(\mathcal{E}, \mathcal{F})}$ on S -schemes, defined by $\mathfrak{H}om_{(\mathcal{E}, \mathcal{F})}(T) = \text{Hom}(\mathcal{E}_T, \mathcal{F}_T)$ where \mathcal{E}_T and \mathcal{F}_T are the pull-backs of \mathcal{E} and \mathcal{F} to $X_T = X \times_S T$. A basic result of Grothendieck ([EGA] III 7.7.8, 7.7.9) says that if \mathcal{F} is flat over S then $\mathfrak{H}om_{(\mathcal{E}, \mathcal{F})}$ is representable for all \mathcal{E} .

We prove the converse of the above, in fact, we show that if L is a relatively ample line bundle on X over S such that the functor $\mathfrak{H}om_{(L^{-n}, \mathcal{F})}$ is representable for infinitely many positive integers n , then \mathcal{F} is flat over S . As a corollary, taking $X = S$, it follows that if \mathcal{F} is a coherent sheaf on S then the functor $T \mapsto H^0(T, \mathcal{F}_T)$ on the category of S -schemes is representable if and only if \mathcal{F} is locally free on S . This answers a question posed by Angelo Vistoli.

The techniques we use involve the proof of flattening stratification, together with the methods used in proving the author's earlier result (see [N1]) that the automorphism group functor of a coherent sheaf on S is representable if and only if the sheaf is locally free.

Let S be a noetherian scheme, and let X be a projective scheme over S . If \mathcal{E} and \mathcal{F} are coherent sheaves on X , consider the contravariant functor $\mathfrak{H}om_{(\mathcal{E}, \mathcal{F})}$ from the category of schemes over S to the category of sets which is defined by putting

$$\mathfrak{H}om_{(\mathcal{E}, \mathcal{F})}(T) = \text{Hom}_{X_T}(\mathcal{E}_T, \mathcal{F}_T)$$

for any S -scheme $T \rightarrow S$, where $X_T = X \times_S T$, and \mathcal{E}_T and \mathcal{F}_T denote the pull-backs of \mathcal{E} and \mathcal{F} under the projection $X_T \rightarrow X$. This functor is clearly a sheaf in the fpqc topology on Sch/S . It was proved by Grothendieck that if \mathcal{F} is flat over S then the above functor is representable (see [EGA] III 7.7.8, 7.7.9).

Our main theorem is as follows, which is a converse to the above.

Theorem 1 *Let S be a noetherian scheme, X a projective scheme over S , and L a relatively very ample line bundle on X over S . Let \mathcal{F} be a coherent sheaf on X . Then the following three statements are equivalent:*

- (1) *The sheaf \mathcal{F} is flat over S .*

- (2) For any coherent sheaf \mathcal{E} on X , the set-valued contravariant functor $\mathfrak{H}om_{(\mathcal{E}, \mathcal{F})}$ on S -schemes, defined by $\mathfrak{H}om_{(\mathcal{E}, \mathcal{F})}(T) = \text{Hom}_{X_T}(\mathcal{E}_T, \mathcal{F}_T)$, is representable.
- (3) There exist infinitely many positive integers r such that the set-valued contravariant functor $\mathcal{G}^{(r)}$ on S -schemes, defined by $\mathcal{G}^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r})$, is representable.

In particular, taking $X = S$ and $L = \mathcal{O}_X$, we get the following corollary.

Corollary 2 *Let S be a noetherian scheme, and \mathcal{F} a coherent sheaf on S . Consider the contravariant functor \mathbf{F} from S -schemes to sets, which is defined by putting $\mathbf{F}(T) = H^0(T, f^*\mathcal{F})$ for any S -scheme $f : T \rightarrow S$. This functor (which is a sheaf in the fpqc topology) is representable if and only if \mathcal{F} is locally free as an \mathcal{O}_S -module.*

Note that the affine line \mathbf{A}_S^1 over a base S admits a ring-scheme structure over S in the obvious way. A **linear scheme** over a scheme S will mean a module-scheme $V \rightarrow S$ under the ring-scheme \mathbf{A}_S^1 . This means V is a commutative group-scheme over S together with a ‘scalar-multiplication’ morphism $\mu : \mathbf{A}_S^1 \times_S V \rightarrow V$ over S , such that the module axioms (in diagrammatic terms) are satisfied.

A **linear functor** \mathbf{F} on S -schemes will mean a contravariant functor from S -schemes to sets together with the structure of an $H^0(T, \mathcal{O}_T)$ -module on $\mathbf{F}(T)$ for each S -scheme T , which is well-behaved under any morphism $f : U \rightarrow T$ of S -schemes in the following sense: $\mathbf{F}(f) : \mathbf{F}(T) \rightarrow \mathbf{F}(U)$ is a homomorphism of the underlying additive groups, and $\mathbf{F}(f)(a \cdot v) = f^*(a) \cdot (\mathbf{F}(f)v)$ for any $a \in H^0(T, \mathcal{O}_T)$ and $v \in \mathbf{F}(T)$. In particular note that the kernel of $\mathbf{F}(f)$ will be an $H^0(T, \mathcal{O}_T)$ -submodule of $\mathbf{F}(T)$. The functor of points of a linear scheme is naturally a linear functor. Conversely, it follows by the Yoneda lemma that if a linear functor \mathbf{F} on S -schemes is representable, then the representing scheme V is naturally a linear scheme over S .

For example, the linear functor $T \mapsto H^0(T, \mathcal{O}_T)^n$ (where $n \geq 0$) is represented by the affine space $\mathbf{A}_{\mathbb{Z}}^n$ over $\text{Spec } \mathbb{Z}$, with its usual linear-scheme structure. More generally, for any coherent sheaf \mathfrak{Q} on S , the scheme $\mathbf{Spec} \text{Sym}(\mathfrak{Q})$ is naturally a linear-scheme over S , where $\text{Sym}(\mathfrak{Q})$ denotes the symmetric algebra of \mathfrak{Q} over \mathcal{O}_S . It represents the linear functor $\mathbf{F}(T) = \text{Hom}(\mathfrak{Q}_T, \mathcal{O}_T)$ where \mathfrak{Q}_T denotes the pull-back of \mathfrak{Q} under $T \rightarrow S$.

With this terminology, the functor $\mathcal{G}^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r})$ of Theorem 1.(3) is a linear functor. Therefore, if a representing scheme $G^{(r)}$ exists, it will naturally be a linear scheme. Note that each $\mathcal{G}^{(r)}$ is obviously a sheaf in the fpqc topology.

The proof of Theorem 1 is by a combination of the result of Grothendieck on the existence of a flattening stratification ([TDTE -IV]) together with the techniques which were employed in [N1] to prove the following result.

Theorem 3 (Representability of the functor GL_E) *Let S be a noetherian scheme, and E a coherent \mathcal{O}_S -module. Let GL_E denote the contrafunctor on S -schemes which associates to any S -scheme $f : T \rightarrow S$ the group of all \mathcal{O}_T -linear*

automorphisms of the pull-back $E_T = f^*E$ (this functor is a sheaf in the fpqc topology). Then GL_E is representable by a group scheme over S if and only if E is locally free.

We re-state Grothendieck's result (see [TDTE IV]) on the existence of a flattening stratification in the following form, which emphasises the role of the direct images $\pi_*(\mathcal{F}(r))$. For an exposition of flattening stratification, see Mumford [M] or [N2].

Theorem 4 (Grothendieck) *Let S be a noetherian scheme, and let \mathcal{F} be a coherent sheaf on \mathbf{P}_S^n where $n \geq 0$. There exists an integer m , and a collection of locally closed subschemes $S_f \subset S$ indexed by polynomials $f \in \mathbb{Q}[\lambda]$, with the following properties.*

(i) *The underlying set of S_f consists of all $s \in S$ such that the Hilbert polynomial of \mathcal{F}_s is f , where \mathcal{F}_s denotes the pull-back of \mathcal{F} to the schematic fibre \mathbf{P}_s^n over s of the projection $\pi : \mathbf{P}_S^n \rightarrow S$. All but finitely many S_f are empty (only finitely many Hilbert polynomials occur). In particular, the S_f are mutually disjoint, and their set-theoretic union is S .*

(ii) *For each $r \geq m$, the higher direct images $R^j\pi_*(\mathcal{F}(r))$ are zero for $j \geq 1$ and the subschemes S_f give the flattening stratification for the direct image $\pi_*(\mathcal{F}(r))$, that is, the morphism $i : \coprod_f S_f \rightarrow S$ induced by the locally closed embeddings $S_f \hookrightarrow S$ has the universal property that for any morphism $g : T \rightarrow S$, the sheaf $g^*\pi_*(\mathcal{F}(r))$ is locally free on T if and only if g factors via $i : \coprod_f S_f \rightarrow S$.*

(iii) *The subschemes S_f give the flattening stratification for \mathcal{F} , that is, for any morphism $g : T \rightarrow S$, the sheaf $\mathcal{F}_T = (1 \times g)^*\mathcal{F}$ on \mathbf{P}_T^n is flat over T if and only if g factors via $i : \coprod_f S_f \rightarrow S$. In particular, \mathcal{F} is flat over S if and only if each S_f is an open subscheme of S .*

(iv) *Let $\mathbb{Q}[\lambda]$ be totally ordered by putting $f_1 < f_2$ if $f_1(p) < f_2(p)$ for all $p \gg 0$. Then the closure of S_f in S is set-theoretically contained in $\bigcup_{g \geq f} S_g$. Moreover, whenever S_f and S_g are non-empty, we have $f < g$ if and only if $f(p) < g(p)$ for all $p \geq m$.*

The following elementary lemma of Grothendieck on base-change does not need any flatness hypothesis. The price paid is that the integer r_0 may depend on ϕ . (See [N2] for a cohomological proof.)

Lemma 5 *Let $\phi : T \rightarrow S$ be a morphism of noetherian schemes, let \mathcal{F} a coherent sheaf on \mathbf{P}_S^n , and let \mathcal{F}_T denote its pull-back under the induced morphism $\mathbf{P}_T^n \rightarrow \mathbf{P}_S^n$. Let $\pi_S : \mathbf{P}_S^n \rightarrow S$ and $\pi_T : \mathbf{P}_T^n \rightarrow T$ denote the projections. Then there exists an integer r_0 such that the base-change homomorphism $\phi^*\pi_{S*}\mathcal{F}(r) \rightarrow \pi_{T*}\mathcal{F}_T(r)$ is an isomorphism for all $r \geq r_0$.*

Proof of Theorem 1 The implication (1) \Rightarrow (2) follows by [EGA] III 7.7.8, 7.7.9, while the implication (2) \Rightarrow (3) follows by taking $\mathcal{E} = L^{\otimes -r}$. Therefore it now remains to show the implication (3) \Rightarrow (1). This we do in a number of steps.

Step 1: Reduction to $S = \operatorname{Spec} R$ with R local, $X = \mathbf{P}_S^n$ and $L = \mathcal{O}_{\mathbf{P}_S^n}(1)$

Suppose that \mathcal{F} is not flat over S , but the linear functor $\mathcal{G}^{(r)}$ on S -schemes, defined by $\mathcal{G}^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r})$, is representable by a linear scheme $G^{(r)}$ over S for arbitrarily large integers r . As \mathcal{F} is not flat, by definition there exists some $x \in X$ such that the stalk \mathcal{F}_x is not a flat module over the local ring $\mathcal{O}_{S, \pi(x)}$ where $\pi : X \rightarrow S$ is the projection. Let $U = \operatorname{Spec} \mathcal{O}_{S, \pi(x)}$, let \mathcal{F}_U be the pull-back of \mathcal{F} to $X_U = X \times_S U$ and let $G_U^{(r)}$ denote the pull-back of $G^{(r)}$ to U . Then \mathcal{F}_U is not flat over U but given any integer m , there exists an integer $r \geq m$ such that the functor $\mathcal{G}_U^{(r)}$ on U -schemes, defined by $\mathcal{G}_U^{(r)}(T) = H^0(X_T, \mathcal{F}_T \otimes L^{\otimes r})$, is representable by the U -scheme $G_U^{(r)}$.

Therefore, by replacing S by U , we can assume that S is of the form $\operatorname{Spec} R$ where R is a noetherian local ring. Let $i : X \hookrightarrow \mathbf{P}_S^n$ be the embedding given by L . Then replacing \mathcal{F} by $i_*\mathcal{F}$, we can further assume that $X = \mathbf{P}_S^n$ and $L = \mathcal{O}_{\mathbf{P}_S^n}(1)$.

Step 2: Flattening stratification of $\operatorname{Spec} R$ There exists an integer m as asserted by Theorem 4, such that for any $r \geq m$, the flattening stratification of S for the sheaf $\pi_*\mathcal{F}(r)$ on S is the same as the flattening stratification of S for the sheaf \mathcal{F} on \mathbf{P}_S^n . Let $r \geq m$ be any integer. As \mathcal{F} is not flat over $S = \operatorname{Spec} R$, the sheaf $\pi_*\mathcal{F}(r)$ is not flat. Let $M_r = H^0(S, \pi_*\mathcal{F}(r))$, which is a finite R -module. Let $\mathfrak{m} \subset R$ be the maximal ideal, and let $k = R/\mathfrak{m}$ the residue field. Let $s \in S = \operatorname{Spec} R$ be the closed point, and let $d = \dim_k(M_r/\mathfrak{m}M_r)$. Then there exists a right-exact sequence of R -modules of the form

$$R^\delta \xrightarrow{\psi} R^d \rightarrow M_r \rightarrow 0$$

Let $I \subset R$ be the ideal formed by the matrix entries of the $(d \times \delta)$ -matrix ψ . Then I defines a closed subscheme $S' \subset S$ which is the flattening stratification of S for M_r . As M_r is not flat by assumption, I is a non-zero proper ideal in R .

It follows from Theorem 4 that I is independent of r as long as $r \geq m$.

Step 3: Reduction to artin local case with principal I with $\mathfrak{m}I = 0$ Let $I = (a_1, \dots, a_t)$ where a_1, \dots, a_t is a minimal set of generators of I . Let $J \subset R$ be the ideal defined by

$$J = (a_2, \dots, a_t) + \mathfrak{m}I$$

Then note that $J \subset I \subset \mathfrak{m}$, and the quotient $R' = R/J$ is an artin local R -algebra with maximal ideal $\mathfrak{m}' = \mathfrak{m}/J$, and $I' = I/J$ is a non-zero principal ideal which satisfies $\mathfrak{m}'I' = 0$. For the base-change under $f : \operatorname{Spec} R' \rightarrow \operatorname{Spec} R$, the flattening stratification for $f^*\pi_*\mathcal{F}(r)$ is defined by the ideal I' for $r \geq m$. Let \mathcal{F}' denote the pull-back of \mathcal{F} to $\mathbf{P}_{R'}^n$, and let $\pi' : \mathbf{P}_{R'}^n \rightarrow \operatorname{Spec} R'$ the projection. As f is a morphism of noetherian schemes, by Lemma 5 there exists some integer m' such that the base-change homomorphism $f^*\pi_*\mathcal{F}(r) \rightarrow \pi'_*\mathcal{F}'(r)$ is an isomorphism whenever $r \geq m'$. Choosing some $m' \geq m$ with this property, and replacing R by R' , \mathcal{F} by \mathcal{F}' and m by m' , we can assume that R is artin local, and I is a non-zero principal ideal with $\mathfrak{m}I = 0$, which defines the flattening stratification for $\pi_*\mathcal{F}(r)$ for all $r \geq m$.

Step 4: Decomposition of $\pi_*\mathcal{F}(r)$ via lemma of Srinivas

Lemma (Srinivas) *Let R be an artin local ring with maximal ideal \mathfrak{m} , and let E be any finite R module whose flattening stratification is defined by an ideal I which is a non-zero proper principal ideal with $\mathfrak{m}I = 0$. Then there exist integers $i \geq 0$ and $j > 0$ such that E is isomorphic to the direct sum $R^i \oplus (R/I)^j$.*

Proof See Lemma 4 in [N1].

We apply the above lemma to the R -module $M_r = H^0(S, \pi_*\mathcal{F}(r))$, which has flattening stratification defined by the principal ideal I with $\mathfrak{m}I = 0$, to conclude that (up to isomorphism) M_r has the form

$$M_r = R^{i(r)} \oplus (R/I)^{j(r)}$$

for non-negative integers $i(r)$ and $j(r)$ with $j(r) > 0$.

Note that $i(r) + j(r) = \Phi(r)$ where Φ is the Hilbert polynomial of \mathcal{F} .

Step 5: Structure of the hypothetical representing scheme $G^{(r)}$ Let $\phi : \text{Spec}(R/I) \hookrightarrow \text{Spec } R$ denote the inclusion and \mathcal{F}' denote the pull-back of \mathcal{F} under $\mathbf{P}_{R/I}^n \hookrightarrow \mathbf{P}_R^n$. The sheaf \mathcal{F}' is flat over R/I , and the functor $\mathcal{G}_{R/I}^{(r)}$, which is the restriction of $\mathcal{G}^{(r)}$, is represented by the linear scheme $\mathbf{A}_{R/I}^d = \text{Spec}(R/I)[y_1, \dots, y_d]$ over R/I , where $d = \Phi(r)$ where Φ is the Hilbert polynomial of \mathcal{F} . Hence, the pull-back of the hypothetical representing scheme $G^{(r)}$ to R/I is the linear scheme $\mathbf{A}_{R/I}^d$. We now use the following fact (see Lemma 6 and Lemma 7 of [N1] for a proof).

Lemma *Let R be a ring and I a nilpotent ideal ($I^n = 0$ for some $n \geq 1$). Let X be a scheme over $\text{Spec } R$, such that the closed subscheme $Y = X \otimes_R (R/I)$ is isomorphic over R/I to $\text{Spec } B$ where B is a finite-type R/I -algebra. Let $b_1, \dots, b_d \in B$ be a set of algebra generators for B over R/I . Then X is isomorphic over R with $\text{Spec } A$ where A is a finite-type R -algebra. Moreover, there exists a set of R -algebra generators a_1, \dots, a_d for A , such that each a_i restricts modulo I to $b_i \in B$ over R/I . Let $R[x_1, \dots, x_d]$ be a polynomial ring in d variables over R , and consider the surjective R -algebra homomorphism $R[x_1, \dots, x_d] \rightarrow A$ defined by sending each x_i to a_i , and let J be its kernel. Then $J \subset IR[x_1, \dots, x_d]$.*

It follows from the above lemma that $G^{(r)}$ is affine of finite type over R , and its co-ordinate ring A as an R algebra is of the form

$$A = R[a_1, \dots, a_d] = R[x_1, \dots, x_d]/J$$

where a_i is the residue of x_i , and a_1, \dots, a_d restrict over R/I to the linear coordinates y_1, \dots, y_d on the linear scheme $\mathbf{A}_{R/I}^d$, and J is an ideal with $J \subset I \cdot R[x_1, \dots, x_d]$. Being an additive group-scheme, $G^{(r)}$ has its zero section $\sigma : \text{Spec } R \rightarrow G^{(r)}$, and this corresponds to an R -algebra homomorphism $\sigma^* : A \rightarrow R$. Modulo I , the section σ restricts to the zero section of $\mathbf{A}_{R/I}^d$ over $\text{Spec}(R/I)$, therefore $\sigma^*(a_i) \in I$ for all $i = 1, \dots, d$. Let $x'_i = x_i - \sigma^*(a_i) \in R[x_1, \dots, x_d]$ and $a'_i = a_i - \sigma^*(a_i) \in A$ be its residue modulo J . Then $R[x_1, \dots, x_d] = R[x'_1, \dots, x'_d]$, the elements a'_1, \dots, a'_d

generate A as an R -algebra, and moreover the a'_i restrict over R/I to the linear coordinates y_i on the linear scheme $\mathbf{A}_{R/I}^d$. Therefore, by replacing the x_i by the x'_i and the a_i by the a'_i , we can assume that for each i , we have

$$\sigma^*(a_i) = 0$$

Next, consider any element $f(x_1, \dots, x_d) \in J$. Then $f(a_1, \dots, a_d) = 0$ in A , so $\sigma^*f(a_1, \dots, a_d) = 0 \in R$, which shows that the constant coefficient of f is zero, as $\sigma^*(a_i) = 0$. As we already know that $J \subset I \cdot R[x_1, \dots, x_d]$, the vanishing of the constant term of any element of J now establishes that

$$J \subset I \cdot (x_1, \dots, x_d)$$

From the above, using $I^2 = 0$, it follows that for any $(b_1, \dots, b_d) \in I^d$, we have a well-defined R -algebra homomorphism

$$\Psi_{(b_1, \dots, b_d)} : A \rightarrow R : a_i \mapsto b_i$$

We now express the linear-scheme structure of $G^{(r)}$ in terms of the ring A , using the fact that each a_i restricts to y_i modulo I , and $G_{R/I}^{(r)}$ is the standard linear-scheme $\mathbf{A}_{R/I}^d$ with linear co-ordinates y_i . Note that the vector addition morphism $\mathbf{A}_{R/I}^d \times_{R/I} \mathbf{A}_{R/I}^d \rightarrow \mathbf{A}_{R/I}^d$ corresponds to the R/I -algebra homomorphism

$$(R/I)[y_1, \dots, y_d] \rightarrow (R/I)[y_1, \dots, y_d] \otimes_{R/I} (R/I)[y_1, \dots, y_d] : y_i \mapsto y_i \otimes 1 + 1 \otimes y_i$$

while the scalar-multiplication morphism $\mathbf{A}_{R/I}^1 \times_{R/I} \mathbf{A}_{R/I}^d \rightarrow \mathbf{A}_{R/I}^d$ corresponds to the R/I -algebra homomorphism

$$(R/I)[y_1, \dots, y_d] \rightarrow (R/I)[t, y_1, \dots, y_d] = (R/I)[t] \otimes_{R/I} (R/I)[y_1, \dots, y_d] : y_i \mapsto ty_i$$

It follows that the addition morphism $\alpha : G^{(r)} \times_R G^{(r)} \rightarrow G^{(r)}$ corresponds to an algebra homomorphism $\alpha^* : A \rightarrow A \otimes_R A$ which has the form

$$a_i \mapsto a_i \otimes 1 + 1 \otimes a_i + u_i \text{ where } u_i \in I(A \otimes_R A).$$

Let the element u_i in the above equation for $\alpha^*(a_i)$ be written as a polynomial expression

$$u_i = f_i(a_1 \otimes 1, \dots, a_d \otimes 1, 1 \otimes a_1, \dots, 1 \otimes a_d)$$

with coefficients in I . The additive identity 0 of $G^{(r)}(R)$ corresponds to $\sigma^* : A \rightarrow R$ with $\sigma^*(a_i) = 0$, and we have $0+0=0$ in $G^{(r)}(R)$. This implies that $f_i(0, \dots, 0) = 0$, and so the constant term of f_i is zero. From this, using $I^2 = 0$, we get the important consequence that

$$f_i(w_1, \dots, w_{2d}) = 0 \text{ for all } w_1, \dots, w_{2d} \in I$$

The scalar-multiplication morphism $\mu : \mathbf{A}_R^1 \times_R G^{(r)} \rightarrow G^{(r)}$ prolongs the standard scalar multiplication on $\mathbf{A}_{R/I}^d$, and so μ corresponds to an algebra homomorphism $\mu^* : A \rightarrow A[t] = R[t] \otimes_R A$ which has the form

$$a_i \mapsto ta_i + v_i \text{ where } v_i \in IA[t].$$

Let v_i be expressed as a polynomial $v_i = g_i(t, a_1, \dots, a_d)$ with coefficients in I . As multiplication by the scalar 0 is the zero morphism on $G^{(r)}$, it follows by specialising under $t \mapsto 0$ that $g_i(0, a_1, \dots, a_d) = 0$. This means $v_i = g_i(t, a_1, \dots, a_d)$ can be expanded as a finite sum

$$v_i = \sum_{j \geq 1} t^j h_{i,j}(a_1, \dots, a_d)$$

where the $h_{i,j}(a_1, \dots, a_d)$ are polynomial expressions with coefficients in I . As the zero vector times any scalar is zero, it follows by specialising under σ^* that $g_i(t, 0, \dots, 0) = 0$. It follows that the constant term of each $h_{i,j}$ is zero. From this, and the fact that $I^2 = 0$, we get the important consequence that

$$g_i(t, b_1, \dots, b_d) = 0 \text{ for all } b_1, \dots, b_d \in I$$

Step 6: The kernel of the map $G^{(r)}(R) \rightarrow G^{(r)}(R/I)$

Lemma *Let $\Psi_{(b_1, \dots, b_d)} : A \rightarrow R$ be the R -algebra homomorphism defined in terms of the generators by $\Psi_{(b_1, \dots, b_d)}(a_k) = b_k$. Let $\Psi : I^d \rightarrow \text{Hom}_{R\text{-alg}}(A, R)$ be the set-map defined by $(b_1, \dots, b_d) \mapsto (\Psi_{(b_1, \dots, b_d)} : A \rightarrow R)$. Then Ψ is a homomorphism of R -modules, where the R -module structure on $\text{Hom}_{R\text{-alg}}(A, R)$ is defined by its identification with the R -module $G^{(r)}(R)$.*

The map Ψ is injective, and its image is the R -submodule $\ker G^{(r)}(\phi) \subset G^{(r)}(R)$, where $\phi : \text{Spec}(R/I) \hookrightarrow \text{Spec } R$ is the inclusion.

Proof For any (b_1, \dots, b_d) and (c_1, \dots, c_d) in I^d , we have

$$\begin{aligned} & (\Psi_{(b_1, \dots, b_d)} + \Psi_{(c_1, \dots, c_d)})(a_i) \\ = & (\Psi_{(b_1, \dots, b_d)} \otimes \Psi_{(c_1, \dots, c_d)})(\alpha^*(a_i)) \\ = & b_i + c_i + f_i(b_1, \dots, b_d, c_1, \dots, c_d) \text{ by substituting for } \alpha^*(a_i) \\ = & b_i + c_i \text{ as } b_k, c_k \in I \\ = & \Psi_{(b_1 + c_1, \dots, b_d + c_d)}(a_i). \end{aligned}$$

This shows the equality $\Psi_{(b_1, \dots, b_d)} + \Psi_{(c_1, \dots, c_d)} = \Psi_{(b_1, \dots, b_d) + (c_1, \dots, c_d)}$, which means the map $\Psi : I^d \rightarrow G^{(r)}(R)$ is additive.

For any $\lambda \in R$, let $f_\lambda : R[t] \rightarrow R$ be the R -algebra homomorphism defined by $f_\lambda(t) = \lambda$. Then for any $(b_1, \dots, b_d) \in I^d$ we have

$$\begin{aligned} (\lambda \cdot \Psi_{(b_1, \dots, b_d)})(a_i) &= (f_\lambda \otimes \Psi_{(b_1, \dots, b_d)})(\mu^*(a_i)) \\ &= (f_\lambda \otimes \Psi_{(b_1, \dots, b_d)})(ta_i + g_i(t, a_1, \dots, a_d)) \\ &= \lambda b_i + g_i(\lambda, b_1, \dots, b_d) \\ &= \lambda b_i \text{ as } b_k \in I \\ &= \Psi_{(\lambda b_1, \dots, \lambda b_d)}(a_i). \end{aligned}$$

This shows the equality $\lambda \cdot \Psi_{(b_1, \dots, b_d)} = \Psi_{\lambda \cdot (b_1, \dots, b_d)}$, hence the map $\Psi : I^d \rightarrow G^{(r)}(R)$ preserves scalar multiplication. This completes the proof that $\Psi : I^d \rightarrow G^{(r)}(R)$ is a homomorphism of R -modules.

The map Ψ is clearly injective. The map $G^{(r)}(\phi) : G^{(r)}(R) \rightarrow G^{(r)}(R/I)$ is in algebraic terms the map $\text{Hom}_{R\text{-alg}}(A, R) \rightarrow \text{Hom}_{R\text{-alg}}(A, R/I)$ induced by the quotient $R \rightarrow R/I$. An element $g \in \text{Hom}_{R\text{-alg}}(A, R/I)$ represents the zero element of $G^{(r)}(R/I)$ exactly when $g(a_i) = 0 \in R/I$ for the generators a_i of A . Therefore $f \in \text{Hom}_{R\text{-alg}}(A, R)$ is in the kernel of $G^{(r)}(\phi)$ precisely when $f(a_i) \in I$ for the generators a_i . Putting $b_i = f(a_i)$, we see that such an f is the same as $\Psi_{(b_1, \dots, b_d)}$.

This completes the proof of the Lemma that $\ker G^{(r)}(\phi) = I^d$.

In particular, as $\mathfrak{m}I = 0$, it follows from the above Lemma that $\ker G^{(r)}(\phi)$ is annihilated by \mathfrak{m} , so it is a vector space over R/\mathfrak{m} , and its dimension as a vector space over R/\mathfrak{m} is $d = \Phi(r)$, as by assumption I is a non-zero principal ideal.

The above determination of the dimension over R/\mathfrak{m} of the kernel of $G^{(r)}(\phi)$ will contradict a more direct functorial description, which is as follows.

Step 7: Functorial description of kernel of $\mathcal{G}^{(r)}(R) \rightarrow \mathcal{G}^{(r)}(R/I)$ As $\mathcal{F}_{R/I}(r)$ is flat over R/I , and as for $r \geq m$ all higher direct images of $\mathcal{F}(r)$ vanish, $\mathcal{G}^{(r)}(R/I)$ is isomorphic to the R/I -module $(R/I)^d$ where $d = \Phi(r)$. By Lemma 5, there exists $m'' \geq m$ such that for $r \geq m''$ the inclusion $\phi : \text{Spec}(R/I) \hookrightarrow \text{Spec } R$ induces an isomorphism $\phi^* \pi_* \mathcal{F}(r) \rightarrow \pi'_* \mathcal{F}'(r)$ where $\pi' : \mathbf{P}_{R/I}^n \rightarrow \text{Spec}(R/I)$ is the projection and \mathcal{F}' is the pull-back of \mathcal{F} under $\mathbf{P}_{R/I}^n \hookrightarrow \mathbf{P}_R^n$. Note that $\mathcal{G}^{(r)}(R) = R^{i(r)} \oplus (R/I)^{j(r)}$, and so for $r \geq m''$ we get an induced decomposition

$$\mathcal{G}^{(r)}(R/I) = (R/I)^{i(r)} \oplus (R/I)^{j(r)}$$

such that the map $\mathcal{G}^{(r)}(\phi) : \mathcal{G}^{(r)}(R) \rightarrow \mathcal{G}^{(r)}(R/I)$ is the map

$$(q, 1) : R^{i(r)} \oplus (R/I)^{j(r)} \rightarrow (R/I)^{i(r)} \oplus (R/I)^{j(r)}$$

where q is the quotient map modulo I . It follows that the kernel of $\mathcal{G}^{(r)}(\phi)$ is the R -module $I^{i(r)} \oplus 0 \subset R^{i(r)} \oplus (R/I)^{j(r)} = \mathcal{G}^{(r)}(R)$. This is a vector space over R/\mathfrak{m} of dimension $i(r) < i(r) + j(r) = \Phi(r)$.

We thus obtain two different values for the dimension of the same vector space $\ker G^{(r)}(\phi) = \ker \mathcal{G}^{(r)}(\phi)$, which shows that our assumption that $\mathcal{G}^{(r)}$ is representable for arbitrarily large values of r is false. This completes the proof of the Theorem 1.

Acknowledgement This note was inspired by a question posed by Angelo Vistoli to the participants of the workshop ‘Advanced Basic Algebraic Geometry’ held at the Abdus Salam ICTP, Trieste, in July 2003. The Corollary 2 answers that question. I thank the ICTP for hospitality while this work was in progress.

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School of Mathematics,
Tata Institute of Fundamental Research,
Homi Bhabha Road,
Mumbai 400 005,
India.
e-mail: nitsure@math.tifr.res.in

05 August 2003